1F

27/7/25

3. $y = x^{1/n}$

$$\frac{d}{dx} y = \frac{d}{dx} x^{1/n}$$

$$= \frac{1}{n} x^{\frac{1-n}{n}}$$

5. $\sin x + \sin y = \frac{1}{2}$

$$\frac{d}{dx}(\sin x + \sin y) = \frac{d}{dx}\left(\frac{1}{2}\right)$$

$$\cos x + \cos y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cos x}{\cos y} = 0$$

$\Rightarrow \cos x = 0$

$$x = \frac{\pi}{2} + k\pi$$

$$\sin y = \frac{1}{2} - \sin\left(\frac{\pi}{2} + k\pi\right)$$

When $k = 2n$ where $n \in \mathbb{Z}$,

$\sin\left(\frac{\pi}{2} + k\pi\right) = 1 \Rightarrow \sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6}, \frac{7\pi}{6}$

When $k = 2n-1$ where $n \in \mathbb{Z}$,

$\sin\left(\frac{\pi}{2} + k\pi\right) = -1 \Rightarrow \sin y = \frac{3}{2}$

$\therefore \left(\frac{\pi}{2} + 2n\pi, -\frac{\pi}{6} + 2n\pi\right), \left(\frac{\pi}{2} + 2n\pi, \frac{7\pi}{6} + 2n\pi\right)$ where $n \in \mathbb{Z}$.

8. a) $V = \frac{1}{3}\pi r^2 h$

$\frac{d}{dr} V = \frac{d}{dr}\left(\frac{1}{3}\pi r^2 h\right)$

$\frac{dV}{dr} = \frac{2}{3}\pi r h$

c) $c^2 = a^2 + b^2 - 2ab\cos\theta$

$\frac{d}{d\theta} c^2 = 0 + 0 - \frac{d}{d\theta}(2ab\cos\theta)$

$2c\frac{dc}{d\theta} = 2ab\sin\theta$

$\frac{dc}{d\theta} = \frac{ab}{c}\sin\theta$

1G

28/7/25

4.

$$y = \frac{1}{x+1}$$

$$\frac{dy}{dx} = -1 \cdot (x+1)^{-2}$$

$$\frac{d_n y}{dx^n} = (-1)^n \cdot n! \cdot (x+1)^{-(n+1)}$$

$$= \frac{(-1)^n n!}{(x+1)^{n+1}}$$

5. $y = u(x)v(x)$

a) $y' = u'(x)v(x) + u(x)v'(x)$

$$y'' = u''(x)v(x) + u'(x)v'(x) + u'(x)v'(x) + u(x)v''(x)$$

$$= u''(x)v(x) + 2u'(x)v'(x) + u(x)v''(x)$$

$$y''' = u'''(x)v(x) + u''(x)v'(x) + 2(u''(x)v'(x) + u'(x)v''(x))$$

$$+ u'(x)v''(x) + u(x)v'''(x)$$

$$= u'''(x)v(x) + 3u''(x)v'(x) + 3u'(x)v''(x) + u(x)v'''(x)$$

b) $y^{(1)} = u^{(1)}v + (1)u^{(0)}v^{(1)} = u^{(1)}v + uv^{(1)}$

$y^{(2)} = u^{(2)}v + \binom{2}{1}u^{(1)}v^{(1)} + \binom{2}{2}u^{(0)}v^{(2)}$

$= u^{(2)}v + 2u^{(1)}v^{(1)} + uv^{(2)}$

$y^{(3)} = u^{(3)}v + \binom{3}{1}u^{(2)}v^{(1)} + \binom{3}{2}u^{(1)}v^{(2)} + \binom{3}{3}u^{(0)}v^{(3)}$

$= u^{(3)}v + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + uv^{(3)}$

$y = x^p(1+x)^q$

$$y^{(p+q)} = x^{p(p+q)}(1+x)^q + \binom{p+q}{1}x^{p(p+q-1)}(1+x)^{q(1)} + \binom{p+q}{2}x^{p(p+q-2)}(1+x)^{q(2)} + \dots + x^p(1+x)^{q(p+q)}$$

$$= (p+q)!\,x^{p-(p+q)}(1+x)^q + (p+q)(p+q-1)!\,x^{p-(p+q-1)}q(1+x)^{q-1} + \frac{(p+q)(p+q-1)}{2!}(p+q-2)!\,x^{p-(p+q-2)}q(q-1)(1+x)^{q-2} + \dots + x^p(p+q)!(1+x)^{q-(p+q)}$$

$$= (p+q)!\,x^{-q}(1+x)^q + \frac{(p+q)!}{1!}x^{-(q-1)}q(1+x)^{q-1} + \frac{(p+q)!}{2!}x^{-(q-2)}q(q-1)(1+x)^{q-2} + \dots + x^p(1+x)^{-p}$$

$$= (p+q)!\left(x^{-q}(1+x)^q + \binom{q}{1}x^{-(q-1)}(1+x)^{q-1} + \binom{q}{2}x^{-(q-2)}(1+x)^{q-2} + \dots + x^p(1+x)^{-p}\right)$$

5A

5A-3 Calculate the derivative with respect to x of the following

a) $\sin^{-1}\left(\dfrac{x-1}{x+1}\right)$ b) $\tanh x$

c) $\ln(x+\sqrt{x^2+1})$ d) y such that $\cos y = x$, $0 \le x \le 1$ and $0 \le y \le \pi/2$.

e) $\sin^{-1}(x/a)$ ~~f)~~ $\sin^{-1}(a/x)$

~~g)~~ $\tan^{-1}(x/\sqrt{1-x^2})$ ~~h)~~ $\sin^{-1}\sqrt{1-x}$

f) Let $y = \sin^{-1}(a/x)$.

$$\Rightarrow \sin y = \frac{a}{x}$$

$$\Rightarrow y' = \frac{1}{\cos y} \frac{d}{dx}\left(\frac{a}{x}\right)$$

$$= \frac{1}{\frac{\sqrt{x^2-a^2}}{x}} \cdot -\frac{a}{x^2}$$

$$= \frac{\cancel{x}}{\sqrt{x^2-a^2}} \cdot -\frac{a}{x^{\cancel{2}}}$$

$$= -\frac{a}{x\sqrt{x^2-a^2}}$$

9) Let $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

$$\tan y = \frac{x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan y = \frac{d}{dx}\frac{x}{\sqrt{1-x^2}}$$

$$\sec^2 y \frac{dy}{dx} = \frac{1\sqrt{1-x^2} - x \cdot \frac{1}{2}\frac{1}{\sqrt{1-x^2}}(-2x)}{(\sqrt{1-x^2})^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{(\sqrt{1-x^2})^2 \sec^2 y}$$

$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{\cancel{(\sqrt{1-x^2})^2}\frac{1}{\cancel{(\sqrt{1-x^2})^2}}}$$

$$= \frac{(1-x^2) + x^2}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

h) Let $y = \sin^{-1}(\sqrt{1-x})$.

$$\Rightarrow \sin y = \sqrt{1-x}$$

$$\Rightarrow y' = \frac{1}{\cos y} \frac{d}{dx}(\sqrt{1-x})$$

$$= \frac{1}{\sqrt{x}} \cdot \frac{1}{2} \frac{1}{\sqrt{1-x}}(-1)$$

$$= -\frac{1}{2\sqrt{x(1-x)}}$$

1H-1 The *half-life* λ of a radioactive substance decaying according to the law $y = y_0 e^{-kt}$ is defined to be the time it takes the amount to decrease to $1/2$ of the initial amount y_0.

a) Express the half-life λ in terms of k. (Do this from scratch — don't just plug into formulas given here or elsewhere.)

b) Show using your expression for λ that if at time t_1 the amount is y_1, then at time $t_1 + \lambda$ it will be $y_1/2$, no matter what t_1 is.

28/7/25

a) $y = y_0 e^{-kt}$

$$\frac{1}{2} y_0 = y_0 e^{-k\lambda}$$

$$\Rightarrow \ln e^{-k\lambda} = \ln \frac{1}{2}$$

$$-k\lambda = -\ln 2$$

$$\lambda = \frac{\ln 2}{k}$$

b) $y_1 = y_0 e^{-kt_1}$

$$y = y_0 e^{-k(t_1+\lambda)}$$

$$= y_0 e^{-kt_1} \cdot e^{-k\lambda}$$

$$= y_0 e^{-kt_1} \cdot \left(\frac{1}{e^{\ln 2}}\right)$$

$$= \frac{y_1}{2}$$

1H-2 If a solution containing a heavy concentration of hydrogen ions (i.e., a strong acid) is diluted with an equal volume of water, by approximately how much is its pH changed? (Express $(\mathrm{pH})_{\text{diluted}}$ in terms of $(\mathrm{pH})_{\text{original}}$.)

$$pH = -\log[H^+]$$

$$(pH)_{original} = -\log x$$

$$\begin{aligned}(pH)_{diluted} &= -\log\frac{x}{2} \\ &= -(\log x - \log 2) \\ &= -\log x + \log 2 \\ &= (pH)_{original} + \log 2\end{aligned}$$

1H-3 Solve the following for y:

~~a)~~ $\ln(y+1) + \ln(y-1) = 2x + \ln x$

b) $\log(y+1) = x^2 + \log(y-1)$

c) $2 \ln y = \ln(y+1) + x$

a) $\ln(y+1) + \ln(y-1) = 2x + \ln x$

$\ln(y+1)(y-1) = 2x + \ln x$

$y^2 - 1 = e^{2x} \cdot e^{\ln x}$

$y^2 = xe^{2x} + 1$

$y = \pm\sqrt{xe^{2x} + 1}$

$x > 0,\ y > 1 \Rightarrow y = \sqrt{xe^{2x} + 1}$

1H-5 Solve for x (hint: put $u = e^x$, solve first for u):

a) $\dfrac{e^x + e^{-x}}{e^x - e^{-x}} = y$ ~~b)~~ $y = e^x + e^{-x}$

b) $y = e^x + e^{-x}$

$$e^x y = e^{2x} + 1$$

$$e^{2x} - e^x y + 1 = 0$$

$$e^{2x} - e^x y + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2 + 1 = 0$$

$$\left(e^x - \frac{y}{2}\right)^2 + 1 - \frac{y^4}{4} = 0$$

$$e^x - \frac{y}{2} = \pm\sqrt{\frac{y^4}{4} - 1}$$

$$x = \ln\left(\frac{y}{2} \pm \frac{\sqrt{y^4 - 4}}{2}\right)$$

1I

1I-1 Calculate the derivatives

a) xe^x b) $(2x-1)e^{2x}$ ~~c)~~ e^{-x^2}

~~d)~~ $x\ln x - x$ ~~e)~~ $\ln(x^2)$ ~~f)~~ $(\ln x)^2$

g) $(e^{x^2})^2$ h) x^x i) $(e^x+e^{-x})/2$

j) $(e^x-e^{-x})/2$ k) $\ln(1/x)$ l) $1/\ln x$

~~m)~~ $(1-e^x)/(1+e^x)$

c) $\frac{d}{dx}(e^{-x^2})$

$= e^{-x^2}\cdot(-2x)$

$= -2x\cdot e^{-x^2}$

f) $\frac{d}{dx}(\ln x)^2$

$= 2\ln x\cdot\frac{1}{x}$

$= \frac{2\ln x}{x}$

d) $\frac{d}{dx}(x\ln x - x)$

$= \ln x + x\cdot\frac{1}{x} - 1$

$= \ln x$

m) $\frac{d}{dx}\left(\frac{1-e^x}{1+e^x}\right)$

$= \frac{-e^x(1+e^x)-(1-e^x)e^x}{(1+e^x)^2}$

$= \frac{-e^x-e^{2x}-e^x+e^{2x}}{(1+e^x)^2}$

$= -\frac{2e^x}{(1+e^x)^2}$

e) $\frac{d}{dx}(\ln(x^2))$

$= 2\cdot\frac{1}{x}$

$= \frac{2}{x}$

1I-4 Using $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$, calculate

~~a)~~ $\lim_{n\to\infty}(1+\frac{1}{n})^{3n}$ b) $\lim_{n\to\infty}(1+\frac{2}{n})^{5n}$ c) $\lim_{n\to\infty}(1+\frac{1}{2n})^{5n}$

a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$= \lim_{m\to 0}(1+m)^{\frac{1}{m}}$

$= \lim_{m\to 0} e^{\ln(1+m)^{\frac{1}{m}}}$

$= e^{\lim_{m\to 0}\frac{1}{m}\ln(1+m)}$

$\lim_{m\to 0}\frac{\ln(1+m)-\ln 1}{m}$

$= \frac{d}{dx}\ln x\Big|_{x=1}$

$= \frac{1}{x}\Big|_{x=1}$

$= 1$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$

$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^3$

$= e^3$

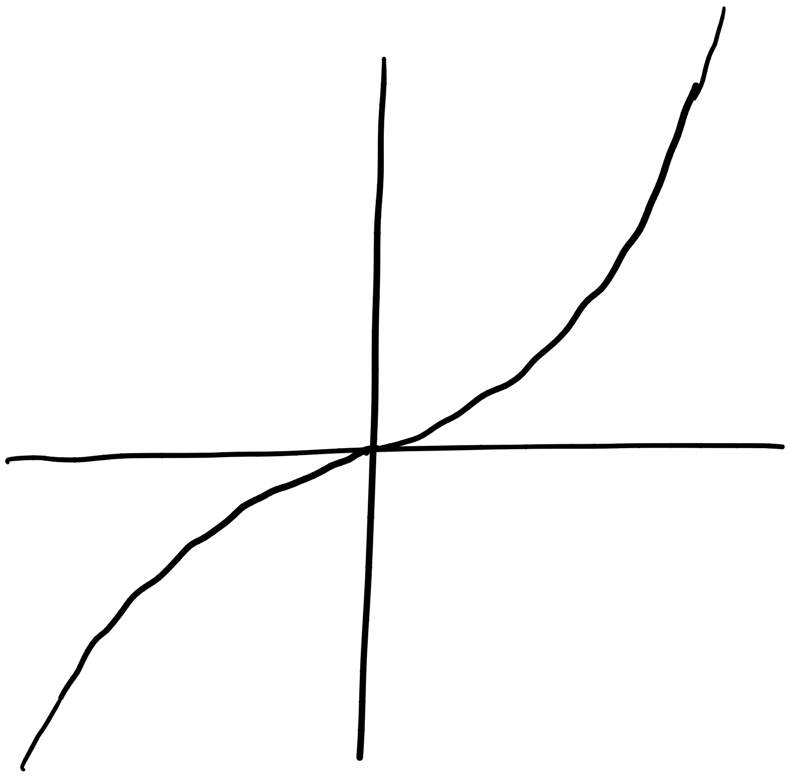
5A

5A-5 a) Sketch the graph of $y = \sinh x$, by finding its critical points, points of inflection, symmetries, and limits as $x \to \infty$ and $-\infty$.

b) Give a suitable definition for $\sinh^{-1} x$, and sketch its graph, indicating the domain of definition. (The inverse hyperbolic sine.)

c) Find $\frac{d}{dx} \sinh^{-1} x$.

a)



$y = \sinh x$

$\frac{dy}{dx} = \cosh x = 0$

$\because \cosh x \neq 0$,

$\therefore$ $\sinh x$ has no critical points.

$\frac{d^2y}{dx^2} = \sinh x = \frac{e^x - e^{-x}}{2}$

$\frac{d^2y}{dx^2} > 0$ when $x > 0$

and $\frac{d^2y}{dx^2} < 0$ when $x < 0$.

At $x=0$, $\sinh x = 0$.

As $x \to \infty$, $\sinh x \to \infty$,

as $x \to -\infty$, $\sinh x \to -\infty$,

$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2}$

$= -\frac{e^x - e^{-x}}{2}$

$= -\sinh x$

$\therefore$ $\sinh x$ is an odd function.

b) $y = \sinh^{-1} x$

$\sinh y = x$

$\frac{e^y - e^{-y}}{2} = x$

$e^y - e^{-y} = 2x$

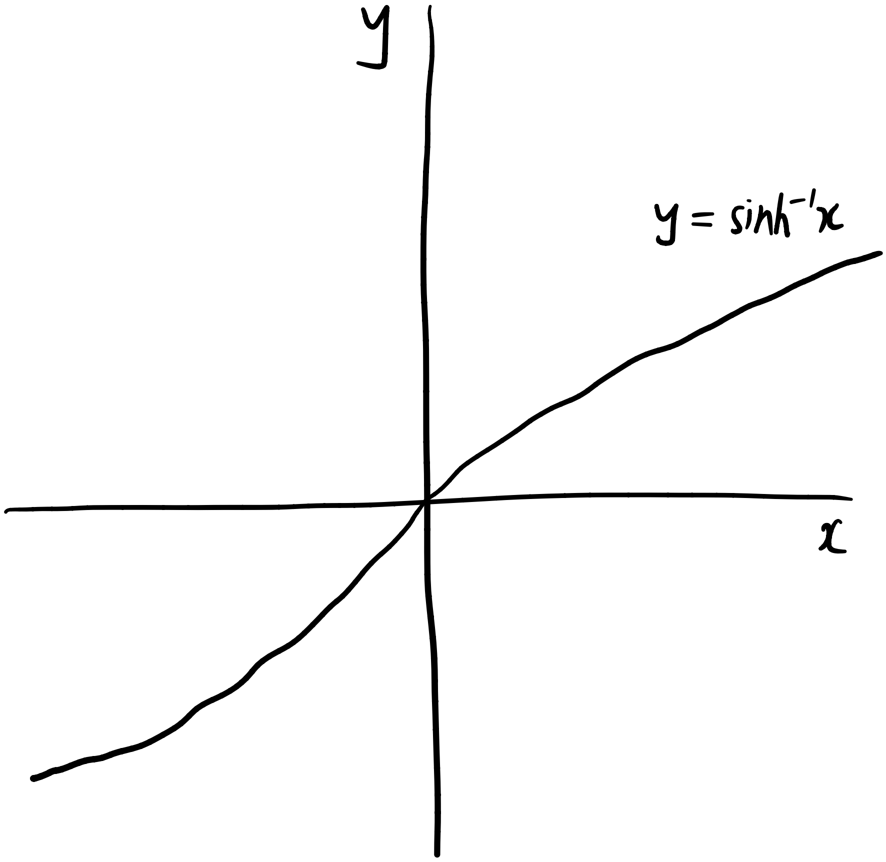
$e^{2y} - 1 = 2xe^y$

$e^{2y} - 2xe^y - 1 + x^2 - x^2 = 0$

$(e^y - x)^2 = x^2 + 1$

$e^y = x + \sqrt{x^2+1}$

$\therefore\ y = \ln(x + \sqrt{x^2+1}),\quad x \in R$



c) $y = \sinh^{-1} x$

$\sinh y = x$

$\frac{d}{dx}(\sinh y) = \frac{d}{dx} x$

$\Rightarrow \cosh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\cosh y}$

$= \frac{1}{\sqrt{\sinh^2 y + 1}}$

$= \frac{1}{\sqrt{x^2+1}}$